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Warm up HW1, MTH 320, Fall 2015

Ayman Badawi

QUESTION 1. a)From now on, we will use the notation D_n to denote the symmetry group of regular n-gon (we will not use S_n, since S_n will mean something else later on, in fact, D_n will be a subgroup of S_n).

b) We will use the notation $(1\ 3)(2\ 4)$ to mean one function (i.e., f(1) = 3, $f(3) = 1, f(2) = 4, f(4) = 2$. However, $(1\ 3)o(2\ 4)$ means a composition of two functions $(1\ 3)$ and $(2\ 4)$. It is easy to see that (1 3)(2 4) = (1 3) o (1 4). Keep this in mind from now and on

(i) Construct the Caley's table of (D_3, o). By staring at the table, convince me it is a group .

(ii) a. Find all elements of D_4. You do not need to construct the Caley's table of D_4. If needed, use the notation $L_{1,2}$ to indicate the perpendicular bisector of the side that connect vertex 1 with vertex 2. You know what L_i should mean!

b. Find $L_4 o L_3$.

c. Find $(2\ 4)o(1\ 3)(2\ 4)$

d. Is $(1\ 4) \in D_4$? Is $(1\ 4)(2\ 3) \in D_4$?

(iii) Submit your solution on Sunday September, 13, 2015.

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Warm up HW II, MTH 320, Fall 2015

Ayman Badawi

QUESTION 1. (i) Given $(A, *)$ is a group. Let $a, b \in A$. Prove that $(a * b)^{-1} = b^{-1} * a^{-1}$.

Proof. Sketch: Since A is associative under $*$, it is clear that $(a * b) * (b^{-1} * a^{-1}) = a * (b * b^{-1}) * a^{-1} = a * e * a^{-1} = e$. Similarly, $(b^{-1} * a^{-1}) * (a * b) = e$. Thus $(a * b)^{-1} = b^{-1} * a^{-1}$.

(ii) Given $(A, *)$ is a group. Let $a, b \in A$ such that $a * b = b * a$. Prove that $a^{-1} * b = b * a^{-1}$. [Hint: note that $a^{-1} * a * b * b^{-1} = e$]

Proof. Sketch: Since $a^{-1}*a*b*b^{-1} = e$ and $a*b = b*a$, we have $a^{-1}*b*a*b^{-1} = e$ implies $(a^{-1}*b*a*b^{-1})*b = eb$ implies $a^{-1} * b * a = b$ implies $(a^{-1} * b * a) * a^{-1} = b * a^{-1}$ implies $a^{-1} * b = b * a^{-1}$

(iii) Let $(A, *)$ be a group such that $a^2 = e$ for every $a \in A$. Prove that A is abelian.

Proof. Sketch: Let $x, y \in A$. By hypothesis, $x^{-1} = x$, $y^{-1} = y$, and $(x * y)^{-1} = x * y$. By part (i), $(x * y)^{-1} = y^{-1} * x^{-1} = y * x$. Thus $x * y = y * x$

(iv) We know that the set of all complex numbers except zero, C^*, under multiplication is group. Let n be a positive integer such that $n \geq 2$. Let $A = \{r \in C^* \mid r^n = 1\}$. Prove that A is a subgroup of C^*. [Hint: Note that A is a finite set]

Proof. Sketch: Since A is finite, it suffices to show that A is closed. Let $x, y \in A$. Hence $(xy)^n = x^n y^n = 1$. Thus $xy \in A$

(v) Let C, D be subgroups of $(F, *)$ such that $C \not\subseteq D$ and $D \not\subseteq C$. Prove that $C \cup D$ is never a subgroup of F.

Proof. Sketch: Deny. Hence $C \cup D$ is a group. By hypothesis, there is an $x \in C - D$ and there is a $y \in D - C$. Since $C \cup D$ is a group, we have $x * y \in C \cup D$. Thus $x * y \in C$ or $x * y \in D$. Assume that $x * y = c \in C$. Thus $y = x^{-1} * c$. Since C is a group and $x, c \in C$, $x^{-1} \in C$ and $x^{-1} * c \in C$. Thus $y = x^{-1} * c \in C$, a contradiction since $y \notin C$ by assumption. Similarly, if $x * y = d \in D$, we can conclude that $x \in D$, a contradiction again. Thus our denial is invalid, and thus $C \cup D$ is never a group.

(vi) Let $A = \{11, 33, 77, 99\}$ and let $* =$ multiplication module 110. Construct the multiplicative Caley's table of $(A, *)$ (you may use calculator to do the calculation.) By staring at the table, it should be clear that $(A, *)$ is a group. Find the identity of A and for each $a \in A$ find a^{-1}.

Typical question, just note that $e = 11$ in this question

(vii) Let $X = \{1, 2, 3\}$. Let $A = P(X)$ be the power set of X (i.e., A is the set of all subsets of X, note that $\phi \in A$ and $X \in A$). Hence A contains exactly 8 elements. Define $*$ on A as : $a * b = (a - b) \cup (b - a)$ for every $a, b \in A$. Construct the Caley's table of $(A, *)$. By staring at the table, it should be clear that $(A, *)$ is a group. Find the identity of A and for each $a \in A$ find a^{-1}. To show associate, just show in general that if $a, b, c \in A$, then $(a * b) * c = a * (b * c)$.

Again, typical question/ just note $e = \phi$

(viii) Let $X = \{1, 2, 3\}$. Let $A = P(X)$ be the power set of X (i.e., A is the set of all subsets of X). Define $*$ on A as : $a * b = a \cap (X - b)$ for every $a, b \in A$. Show that $(A, *)$ has a right identity, say e, and each element in A has a right inverse(i.e., for each $d \in A$, there is a $c \in A$ such that $d * c = e$.). Show that A is NOT a group!!! Does that contradict one of the results we proved in class? Explain

No contradiction, note that A is not associative under $*$

(ix) *Submit your solution in class at 2pm, Tuesday September, 29, 2015*

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HW III, MTH 320, Fall 2015

Ayman Badawi

QUESTION 1. (i) Find the order of each element in $(Z_6, +)$.

(1) it is trivial/ typical question

(ii) Show that (Z_7^*, X) has an element of order 3. Construct a subgroup of (Z_7^*, X) with exactly 3 elements.

(2) it is typical question

(iii) Let $(A, *)$ be a group and assume $a \in A$ such that $|a| = m < \infty$. Let n be a positive integer $n > 0$ such that $gcd(m, n) = 1$. Let $b = a^n$. Then clearly $b \in A$ since A is closed. Prove that $|b| = m$.

proof. It is clear that $b^m = (a^n)^m = (a^m)^n = e$**. Let** $k = |b|$**. Since** $b^m = e$**, we conclude that** $k \mid m$**. We show** $m \mid k$**, and hence** $m = k$**.** $b^k = (a^n)^k = a^{nk} = e$**. Since** $|a| = m$ **and** $a^{nk} = e$**, we know that** $m \mid nk$**. Since** $gcd(m, n) = 1$ **and** $m \mid nk$**, we conclude that** $m \mid k$**. Since** $k \mid m$ **and** $m \mid k$**,** $m = k$**.**

(iv) Let A, B be subgroups of a group F. Prove that $A \cap B$ is a subgroup of F.

Done in class

(v) Let $A = (Z_{12}[X], +)$, the set of all polynomials with coefficient from Z_{12}. We know that A under addition module 12 is an abelian group. Find $|2x|$, $|3|$, $|2x + 3|$. do you observe a relation between the three numbers : $|2x|$, $|3|$, $|2x + 3|$? if yes, what is it?

Typical question

(vi) *Submit your solution in class at 2pm, Thursday, October 8 , 2015*

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HW IV MTH 320, Fall 2015

Ayman Badawi

QUESTION 1. (i) Given $(A, *)$ is a cyclic (finite) group. Prove that A is an abelian group.

(ii) Let $(A, *)$ be a finite group such that $|A| = m$ for some prime integer m. Prove that A is an abelian group. Can you say more about A?

(iii) For a fixed integer n define $U(n) = \{a \mid 1 \leq a < n \text{ and } gcd(a, n) = 1\}$. Prove that (U(12), *) is an abelian group, where $*$ means multiplication module 12.[Hint just construct the Caley's table for $U(12)$.]

(iv) Is the group in part III a cyclic group?

(v) Let A be a cyclic group of order 8, say $A =< a >$. Let $d = a^2, f = a^4, h = a^6, w = a^3$. Find $|d|, |f|, |h|, |w|$.

(vi) **Submit your solution in class at 2pm on Tuesday October 13 or on Wednesday October 14 any time before 4pm (just slide it under my door)**

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HW V MTH 320, Fall 2015

Ayman Badawi

QUESTION 1. (i) Given $(A, *)$ is a group with 72 elements. Assume there exist $a, b \in A$ such that $a * b = b * a$, $|a| = 8$ and $|b| = 9$. Prove that A is an abelian group. Can you say more?

(ii) Let $(A, *)$ be a finite cyclic group with exactly 10 elements, say $A =< a >$ for some $a \in A$. Find all values of k, $1 \leq k \leq 10$ such that $A =< a^k >$.

(iii) Let $(A, *)$ be an abelian group. Assume that $a, b \in A$ such that $|a| = 8$, $|b|$ = 12. Find $|b^4|$, $|a^3|$, $|a * b^4|$.

(iv) Given $(A, *)$ is a group. Assume there exist $a, b \in A$ such that $|a| = n$ and $|b| = m$ such that $gcd(n, m/gcd(n, m)) = 1$. If $a * b = b * a$, prove that there is an element $d \in A$ such that $|d| = LCM[n, m]$. [Hint: Let $f = gcd(n, m)$, and $h = a * b^f$. Then .....]

(v) Let $A = (Z_{12}, +)$, $D = \{0, 3, 6, 9\}$. Then D is a subgroup of A (Do not show that). Find all distinct left cosets of D

(vi) **Submit your solution in class at 2pm on Thursday October 22, 2015)**

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HW 6 MTH 320, Fall 2015

Ayman Badawi

QUESTION 1. (i) Let $(A, *)$ be a finite group of order $m < \infty$. Prove that $a^m = e$ for every $a \in A$ (Trivial, but we need to emphasise this fact).

(ii) Find all subgroups of $(Z_{12}, +)$.

(iii) Show that $(Z, +)$ is a cyclic group. Find all generators of Z.

(iv) Consider the group $(Q, +)$. Show that each non-identity element in $(Q, +)$ has infinite order. Show that every nontrivial (i.e. $\neq \{0\}$) subgroup of $(Q, +)$ is infinite. Show that $(Q, +)$ is not cyclic. [Hint: Assume $Q =< a >$ for some $a \in Q$. Then reach contradiction, for example there is an integer m such that $1 = a^m = ma$ so what is a? now can you find an integer n such that $\frac{1}{m+1} = a^n$?]

(v) Let H be the subgroup of order 2 of $(Z_{10}, +)$. You should know how to construct H. Let L be the set of all left cosets of H. Then L must have exactly 5 elements. Note that each element in L is of the form $d + H$ for some $d \in Z_{12}$ and d need not be unique. Define an operation, say *, on L such that for every $A, B \in L$ $A*B = (a+b)+H$, where $A = a + H, B = b + H$ for some $a, b \in Z_{10}$ (again a, b need not be unique). Show that $(L, *)$ is a cyclic group (note here + means addition module 10). Find an element $v \in L$ such that $L =< v >$. Only construct the Caley's Table for L (Associative is clear, so do not show it, by staring at the table find the identity and the inverse of each element). This problem will help you understand more interesting stuff to come!

(vi) Let Let $(A, *)$ be a finite group of order $m < \infty$. Let n be a positive integer such that $gcd(m, n) = 1$. Let $a \in A$. Prove there is an element $c \in A$ such that $c^n = a$. [This result will be used later on : note that $1 = km + sn$ for some integers $k, s \in Z$, so $a^1 = ...$].

(vii) **Submit your solution in class at 2pm on Thursday October 29, 2015)**

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HW 7 MTH 320, Fall 2015

Ayman Badawi

QUESTION 1. (i) Let $A = \{D \in M_{2\times 2}(Z_5) \mid det(D) \neq 0\}$. Show that A is a group under matrix multiplication. Is $L = \begin{bmatrix} 2 & 4 \\ 1 & 4 \end{bmatrix} \in A$?. If yes, find L^{-1}.

(ii) Let $A = (Z_{17}^*, X)$. Then we know that A is a cyclic group. Find an element a such that $A =< a >$. Show that A has exactly 4 elements each is of order 8.

(iii) Convince me that D_4 (symmetric group of a square) is not cyclic.

(iv) Given $(A, *)$ is a group with 12 elements. Assume there is $a \in A$ such that $a^k \neq a$, for each integer k, $3 \leq k \leq 7$. Prove that A is cyclic.

(v) **Submit your solution in class at 2pm on Thursday Nov 5 OR on Wed Nov 4 at 5:30pm 2015)**

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HW 8 MTH 320, Fall 2015

Ayman Badawi

QUESTION 1. (i) Let $(A, *)$ be an abelian group with 24 elements. Given that H is a subgroup of A with 6 elements. Suppose that $a \in A$ such that $|a| = 3$. Prove that $a \in H$. [Deny. Then show that $a^2 \notin H$ and $a^2 \notin a * H$ as well. Then prove that $H \cup a * H \cup a^2 * H$ is a subgroup of A and reach a contradiction]

(ii) Find the order of the element $(4\ 3\ 2\ 6)o(2\ 3\ 1\ 5) \in S_6$. [Hint first find the element (i.e., the bijection function that is determined from the composition of the two given bijection functions), then use class result].

(iii) Given $(1\ 6\ 3\ 4), (2\ 7\ 5) \in D$, where D is a subgroup of S_7. Prove that D has a subgroup of order 6, and a subgroup of order 2.

(iv) Let $n \geq 2$ and set $A_n = \{a \in S_n \mid \text{a is an even permutation }\}$. Prove that A_n is a subgroup of S_n[This is a very important subgroups!!!]

(v) **Submit your solution in class at 2pm on Sunday November 22, 2015)**

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HW 9 MTH 320, Fall 2015

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QUESTION 1. (i) Let A be a finite cyclic group with n elements and let H be a subgroup of A with m elements. Then:

a. Explain why H is a normal subgroup of A?

b. Prove that A/H is cyclic group. How many elements does A/H have?

c. Assume $A =< a >$ for some $a \in A$. Prove that $A/H =< a * H >$.

(ii) Let A be a group with $37X4X11X15$ elements. Assume $a \in A$ such that $|a| = 74$ and assume that H is a normal subgroup of A with 148 elements. Prove that $a \in H$.[NICE!!!]

(iii) Construct a group homomorphism $F : (Z_{42}, +) \to (Z_6, +)$ that is ONTO (i.e., Image (F) = Z_6) [Hint: Number (i) above might be very useful here]

(iv) Let $F : A \to B$ be a group homomorphism.

a. If H is a subgroup of A, then prove that $F(H) = \{f(h)|h \in H\}$ is a subgroup of Image(F), and if H is normal, then prove that $F(H)$ is normal in Image(F).

Sketch: Let $x, y \in F(H)$**. We show** $x^{-1}y \in F(H)$**. Hince** $x = F(v)$ **and** $y = F(m)$ **for some** $v, m \in H$**. Since** F **is a group homomorphism, we know** $x^{-1} = F(v^{-1}$**. Since** H **is a subgroup of** A**, we know** $v^{-1}m \in H$**. Since F is a group homomorphism and** $v^{-1}m \in H$**, we have** $x^{-1}y = F(v^-1m) \in H$**.**

Now suppose that H **is normal in A. Let** $x \in Image(F)$**. W show** $xF(H) = F(H)x$**. Since** $x \in Image(F)$**,** $x = f(v)$ **for some** $v \in A$**. We know** $vH = Hv$**. Hence** $F(vH) = F(Hv)$**. Thus** $F(v)F(H) = F(H)F(v)$ **and hence** $xF(H) = F(H)x$**.**

b. If M be a subgroup of B, then prove that $L = F^{-1}(M) = \{a \in A|F(a) \in M\}$ is a subgroup of A, and if M is normal, then prove L in normal in A.

Sketch: Let $x, y \in L$**. We show** $x^{-1}y \in L$**. Hence** $F(x) = m_1$**,** $F(y) = m_2 \in M$**. Since** M **is group,** $F(x)^{-1}F(y) = F(x^{-1}y) = d \in M$**. Thus** $x^{-1}y \in L$**.**

Assume M **is normal in** B**. First observe that** $Ker(F) \subseteq L$ **(since** $e_B \in M$ **and** $F^{-1}(e_B) = Ker(F)$**). Let** $x \in A$**. We show** $xL = Lx$**. Let** $k \in L$**. We know that** $xk = ax$ **for some** $a \in A$**. We only need to show that** $a \in L$**. Since** $xk = ax$**, we have** $F(xk) = F(ax)$**. Since** F **is a group homomorphism, we have** $F(xka^{-1}x^{-1}) = e_B$**. Thus** $xka^{-1}x^{-1} = n \in Ker(F)$**. Thus** $a^{-1} = k^{-1}x^{-1}nx$**. Since Ker(F) is normal in A, we have** $x^{-1}nx = g \in Ker(F)$**. Thus** $a^{-1} = k^{-1}g$**. Since** $Ker(F)$ **"lives" inside L and** $k \in L$ **and** $g \in Ker(F)$ **and** L **is a group, we have** $k^{-1}g \in L$**. Hence** $a^{-1} \in L$**. Since** L **is group,** $a \in L$**,**

(v) **Submit your solution in class at 2pm on Thursday December 10, 2015**

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